The 37th International Mathematical Olympiad

First day (10 July, 1996)

Time: $4\frac{1}{2}$ hours   Version: English

1. Let $ABCD$ be a rectangular board with $|AB| = 20$, $|BC| = 12$. The board is divided into 20×12 unit squares. Let r be a given positive integer. A coin can be moved from one square to another if and only if the distance between the centres of the two squares is $\sqrt{r}$. The task is to find a sequence of moves taking the coin from the square which has A as a vertex to the square which has B as a vertex.

   (a) Show that the task cannot be done if r is divisible by 2 or 3.

   (b) Prove that the task can be done if $r = 73$.

   (c) Can the task be done when $r = 97$?

2. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

   Let D, E be the incentres of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.

3. Let $\mathbf{S} = \{0, 1, 2, 3, \ldots\}$ be the set of non-negative integers. Find all functions f defined on $\mathbf{S}$ and taking their values in $\mathbf{S}$ such that

$$f(m + f(n)) = f(f(m)) + f(n) \text{ for all } m, n \text{ in } \mathbf{S}.$$

Each problem is worth 7 points.

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Second day (11 July, 1996)

Time: $4\frac{1}{2}$ hours | Version: English

4. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. Find the least possible value that can be taken by the minimum of these two squares.

5. Let $ABCDEF$ be a convex hexagon such that AB is parallel to ED, BC is parallel to FE and CD is parallel to AF. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF respectively, and let p denote the perimeter of the hexagon. Prove that
$$R_A + R_C + R_E \geq \frac{p}{2}.$$

6. Let n, p, q be positive integers with $n > p + q$. Let $x_0, x_1, \ldots, x_n$ be integers satisfying the following conditions:

   (a) $x_0 = x_n = 0$;

   (b) for each integer i with $1 \leq i \leq n$,
   $$\text{either } x_i - x_{i-1} = p \text{ or } x_i - x_{i-1} = -q.$$

   Show that there exists a pair (i, j) of indices with $i < j$ and $(i, j) \neq (0, n)$ such that $x_i = x_j$.

Each problem is worth 7 points.