37th International Mathematical Olympiad

Solutions

Problem 1

We shall work on the array $\mathcal{A}$ of lattice points defined by

$$\mathcal{A} = \{(i,j) \in \mathbf{Z}^2 : 0 \le i \le 19, 0 \le j \le 11\}.$$

Our task is to move from $(0,0)$ to $(19,0)$ via the points of $\mathcal{A}$ such that each move has length $\sqrt{r}$. Thus for each move of the form $(x,y) \to (x+a, y+b)$, we must have $a^2 + b^2 = r$.

(a) If r is even, then for each solution (a,b) of $a^2+b^2=r$, the sum $a+b$ is even, so for each lattice point (x,y) reached from $(0,0)$, the parity of $x+y$ must be the same as that of $0+0$; that is, $x+y$ must be even. It follows that $(19,0)$ cannot be reached from $(0,0)$.

If r is a multiple of 3, then for each solution (a,b) of $a^2+b^2=r$, both a and b must be multiples of 3; this holds because -1 is not a square modulo 3. Thus for each lattice point (x,y) reached from $(0,0)$, x and y must both be multiples of 3, and so in this case too $(19,0)$ cannot be reached from $(0,0)$.

(b) Consider the case $r = 73 = 8^2 + 3^2$. Let a, b, c and d represent, respectively, the number of moves of the types $\pm(8,3)$, $\pm(8,-3)$, $\pm(3,8)$ and $\pm(3,-8)$. (More precisely, a is the number of moves of type $(8,3)$ minus the number of moves of type $(-8,-3)$; similarly for the others.) Since we have to reach $(19,0)$ from $(0,0)$, we have

$$8(a+b) + 3(c+d) = 19, \quad 3(a-b) + 8(c-d) = 0.$$

Taking $(a+b, c+d) = (2,1)$ as a trial solution of the first equation, and $(a-b, c-d) = (-8,3)$ as a trial solution of the second, we find that

$$a = -3, b = 5, c = 2, d = -1.$$

We now attempt a solution with three moves of type $(-8,-3)$, five moves of type $(8,-3)$, two moves of type $(3,8)$ and one of type $(-3,8)$. The constraint is that we must keep within the boundaries of the array. After some experimentation, the following route emerges:

$(0,0) \to (8,3) \to (11,5) \to (19,2) \to (16,10) \to (8,7) \to (0,4) \to (8,1) \to (11,9) \to (3,6) \to (11,3) \to (19,0)$.

Note that the solution $(a+b, c+d) = (2,1), (a-b, c-d) = (8,-3)$, which gives $a = 5, b = 3, c = 1$ and $d = 2$, also yields a route:

$(0,0) \to (8,3) \to (16,6) \to (8,9) \to (5,1) \to (13,4) \to (5,7) \to (13,10) \to (16,2) \to (8,5) \to (16,8) \to (19,0)$.

(c) If $r = 97$, then since the only way of writing 97 as the sum of two squares is $97 = 9^2 + 4^2$, each of the moves must consist of one of the vectors $(\pm 9, \pm 4)$, $(\pm 4, \pm 9)$. Let the points of $\mathcal{A}$ be partitioned as $\mathcal{B} \cup \mathcal{C}$ in the collowing manner:

$$\mathcal{B} = \{(i,j) \in \mathbf{Z}^2 : 0 \le i \le 19, 4 \le j \le 7\}, \quad \mathcal{C} = \mathcal{A} \setminus \mathcal{B}.$$

Then it can be verified that moves of the type $(\pm 9, \pm 4)$ always take us from points in $\mathcal{B}$ to points in $\mathcal{C}$ and *vice versa*, while moves of type $(\pm 4, \pm 9)$ always take us from points in $\mathcal{C}$ to points in $\mathcal{C}$. (Note that it is not possible to go from one point in $\mathcal{B}$ to another point in $\mathcal{B}$ in one step.)

Each move of the type $(\pm 9, \pm 4)$ changes the parity of the x-coordinate, so since we have to go from $(0,0)$ to $(19,0)$, and *odd* number of such moves is required. Each such move takes us from $\mathcal{B}$ to $\mathcal{C}$ or *vice versa*, so since the starting point $(0,0)$ is in $\mathcal{C}$, we shall end up at a point in $\mathcal{B}$. However, $(19,0) \in \mathcal{C}$. It follows that the required sequence of moves does not exist.

Problem 2

Lemma: Let the feet of the perpendiculars from P to BC, CA and AB be X, Y and Z respectively. Then (i) $YZ = PA \sin A$ (ii) $\angle YXZ = \angle BPC - \angle A$.

This is easy to see via an examination of the three cyclic quadrilaterals $AZPY$, $BXPZ$ and $CYPX$.

Let BD and CE meet AP in Q and R respectively. By the angle bisector theorem, $AQ/QP = AB/BP$ and $AR/RC = AC/CP$. To show that Q, R coincide, it suffices to show that $AB/BP = AC/CP$. Now,

$$\frac{AB}{BP} = \frac{AC}{CP} \iff AB \cdot CP = AC \cdot BP \iff CP \cdot \sin C = BP \cdot \sin B$$
$$\iff XY = XZ \quad \text{(using the Lemma)}.$$

But we are given that $\angle APB - \angle C = \angle APC - \angle B$. This implies that $\angle XZY = \angle XYZ$ (also by the Lemma), so $XY = XZ$ as desired.

Alternative solution: The problem requires us to prove that the internal bisectors of $\angle ABP$ and $\angle ACP$ meet an AP. Let Γ denote the circumcircle of ABC, and let AP, BP and CP meet Γ in X, Y and Z respectively. The condition $\angle APB - \angle C = \angle APC - \angle B$, is equivalent to $\angle PAC + \angle PBC = \angle PAB + \angle PCB$, which simplifies to $\angle XZY = \angle XYZ$ and therefore to $XY = XZ$. Now $BPS \sim ZPY$, so $BC/ZY = BP/ZP = PC/PY$. Let $BP \cdot PY = k$; then $PC \cdot ZP = k = AP \cdot PX$. From the relation

$$\frac{BC}{ZY} = \frac{BP}{ZP} = \frac{BP}{k/PC} = \frac{BP \cdot PC}{k},$$

we deduce that

$$YZ = k\left(\frac{BC}{BP \cdot CP}\right).$$

Similarly

$$XY = k\left(\frac{AB}{AP \cdot BP}\right), \quad XZ = k\left(\frac{AC}{AP \cdot CP}\right).$$

Since $XY = XZ$, it follows that

$$\frac{AB}{BP} = \frac{AC}{CP},$$

that is, $BA/BP = CA/CP$. Thus the internal angle bisector $\angle ABP$ and $\angle ACP$ divide AP in the same ratio, and therefore meet on AP. The stated result follows.

Remark: The locus of P subject to the stated condition is an arc of a circle, because the condition is equivalent to $PB/PC = c/b$.

Problem 3

Putting $m = n = 0$ we obtain $f(0) = 0$ and hence $f(f(n)) = f(n)$ for all $n \in \mathbf{N}_0$. Thus the given functional equation is equivalent to

$$f(m + f(n)) = f(m) + f(n), \quad f(0) = 0.$$

We also observe that if $f(x)$ is not the zero function then f has non-zero fixed points. Let a be the least non-zero fixed point of f. If $a = 1$ then it is easy to check that $f(2) = 2$ and by induction that $f(n) = n$ for all $n \in \mathbf{N}_0$.

Suppose $a > 0$. Again by induction $f(ka) = ka$ for all $k \geq 1$. We shall show that the fixed points of f are all of the form ka for some $k \geq 1$. First note that the sum of two fixed points of f is itself a fixed point. Let b be an arbitrary fixed point of f. Choose non-negative integers q, r such that $b = aq + r, 0 \leq r < a$. Then we get

$$b = f(b) = f(aq + r) = f(r + f(aq)) = f(r) + f(aq) = f(r) + aq.$$

It follows that $f(r) = r$ and since $r < a$ we must have $r = 0$. This proves the claim that the fixed points are all of the form ka. Since the set $\{f(n) : n \in \mathbf{N}_0\}$ is a set of fixed points of f it follows in particular that $f(i) = an_i$ for each $i < a$, with $n_0 = 0$ and $n_i \in \mathbf{N}_0$.

Take any positive integer n and write it as $n = ka + i$ where $0 \leq i < a$. Using the functional equation we obtain

$$f(n) = f(i + ka) = f(i + f(ka)) = f(i) + ka = n_i a + ka = (n_i + k)a.$$

We verify that such an f satisfies the given functional equation: take $m = ka + i, n = la + j, 0 \leq i, j < a$. Then

$$\begin{aligned} f(m+f(n)) = f(ka+i+f(la+j)) &= f((k+l+n_j)a+i) \\ &= (k+l+n_j+n_i)a \\ &= f(m)+f(n) \end{aligned}$$

Thus if f is not identically zero, then f has the following general form: let $a \in \mathbf{N}$ and $n_1, n_2, \ldots, n_{a-1} \in \mathbf{N}_0$ be chosen arbitrarily; then

$$f(n) = \left(\left\lfloor \frac{n}{a} \right\rfloor + n_i \right) a.$$

Problem 4

Let $15a + 16b = r^2, 16a - 15b = s^2$, where $r, s \in \mathbf{N}$. We now obtain:

$$r^4 + s^4 = (15^2 + 16^2)(a^2 + b^2) = 481(a^2 + b^2).$$

Note that $481 = 13 \times 37$. We now use the fact that -1 is not a fourth power either modulo 13 or modulo 37. (To see why this holds, note that the congurence $-1 \equiv x^4 \pmod{13}$ for some $x \in \mathbf{N}$ leads via Fermat's theorem, to $(-1)^3 \equiv 1 \pmod{13}$, which is false; likewise, the congurence $-1 \equiv x^4 \pmod{37}$ for some $x \in \mathbf{N}$ leads to $(-1)^9 \equiv 1 \pmod{37}$, which too is false.)

Since $r^4 + s^4 \equiv 0 \pmod{13}$, either $r \equiv s \equiv 0 \pmod{13}$ or $r \not\equiv 0, s \not\equiv 0$ (both modulo 13). The latter possibility cannot occur because -1 is not a fourth power modulo 13; therefore $r \equiv s \equiv 0 \pmod{13}$, and similarly $r \equiv s \equiv 0 \pmod{37}$. Therefore r and s are both multiples of 481, and so $r \geq 481, s \geq 481$. It is easy to chech that $r = s = 481$ is realizable: we obtain

$$a = 481 \cdot 31, \quad b = 481.$$

Thus the required answer is 481^2.

Problem 5

Let a, b, c, d, e and f denote the lengths of the sides AB, BC, DE, EF and FA respectively. Note that the opposite angles of the hexagon are equal ($\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$). Draw perpendiculars as follows: $AP \perp BC$, $AS \perp EF$, $DQ \perp EF$. Then $PQRS$ is a rectangle and $BF \geq PS = QR$. Therefore $2BF \geq PS + QR$, and so

$$2BF \geq (a \sin B + f \sin C) + (c \sin C + d \sin B).$$

Similarly,

$$2DB \geq (c \sin A + b \sin B) + (e \sin B + f \sin A),$$
$$2FD \geq (e \sin C + d \sin A) + (a \sin A + b \sin C).$$

Next, the circumradii of the triangles FAB, BCD and DEF are related to BF, DB and FD as follows:

$$R_A = \frac{BF}{2\sin A}, \quad R_C = \frac{DB}{2\sin C}, \quad R_E = \frac{FD}{2\sin B}.$$

We obtain, therefore,

$$\begin{aligned} 4(R_A + R_C + R_E) &\geq a\left(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B}\right) + b\left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B}\right) + \cdots \\ &\geq 2(a + b + \cdots) = 2P, \end{aligned}$$

and so $R_A + R_B + R_C \geq P/2$, as required. Equality holds iff $\angle A = \angle B = \angle C$ and $BF \perp BC, \ldots$; that is, iff the hexagon is regular.

Problem 6

We first remark that there is no loss in taking p and q to be coprime; for, if p, q have a common factor $d > 1$, then we can reword the problem in terms of the quantities $p' = p/d$, $q' = q/d$, $x_i' = x_i/d$.

Let there be k indices $i \in \{1, 2, \ldots, n\}$ such that $x_i - x_{i-1} = p$; then the number of indices $i \in \{1, 2, \ldots, n\}$ such that $x_i - x_{i-1} = -q$ is $n-k$. Since $x_n = x_0 = 0$, we see that $kp = (n-k)q$, and since p, q are coprime this implies that $k = aq, n-k = ap$ for some positive integer a. It follows that $n = a(p+q)$, and since $n > p+q$, we have $a > 1$.

Let $y_i = x_{i+p+q} - x_i$ for $i \in \{0, 1, \ldots, n-p-q\}$. Since $n > p+q$, there is more than one y_i. We shal show that at least one of the y_i is 0, which will establish the stated claim. (In fact, this establishes a stronger statement.)

For each i, let S_i denote the set of indices $\{i+1, i+2, \ldots, i+p+q\}$. Let r be the number of $j \in S_i$ for which $x_j - x_{j-1} = p$; then the number of $j \in S_j$ for which $x_j - x_{j-1} = -q$ is $p+q-r$. Summing these equalities over all $j \in S_i$, we obtain

$$y_i = rp - (p+q-r)q = (p+q)(r-q).$$

Thus y_i is a multiple of $(p+q)$ for each i. Now consider the expression $y_{i+1} - y_i$:

$$\begin{aligned} y_{i+1} - y_i &= (x_{i+p+q+1} - x_{i+1}) - (x_{i+p+q} - x_i) \\ &= (x_{i+p+q+1} - x_{i+p+q}) - (x_{i+1} - x_i) \end{aligned}$$

Since each bracketed term is p or $-q$, it follows that $y_{i+1} - y_i$ is 0 or $pm(p+q)$. Next, consider the relation:

$$y_0 + y_{p+q} + y_{2(p+q)} + \cdots + y_{n-p-q} = 0.$$

This shows that the y_i's are neither all positive or all negative. Thus in the sequence

$$y_0, y_1, y_2, \ldots, y_{n-p-q-1}, y_{n-p-q},$$

there exists two adjacent y's that are not of the same sign. Since each y_i is a multiple of $(p+q)$, and since the difference between adjacent y_i's is always 0 or $\pm(p+q)$, it follows that some y_i equals 0.