English

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In the final round of the Abel contest there are 4 problems (8 subproblems) to be solved in 4 hours. You are required to show the reasoning behind your answers. Start a new sheet of paper for each problem.

The maximum score is 10 points for each problem. The total score is thus between 0 and 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

We consider the sum of the digits of a positive integer. For example, the sum of the digits of 2007 is equal to 9, since $2 + 0 + 0 + 7 = 9$.

(a) How many integers n, where $0 < n < 100\,000$, have an even sum of digits?

(b) How many integers n, where $0 < n < 100\,000$, have a sum of digits that is less than or equal to 22?

Problem 2

The vertices of a convex pentagon $ABCDE$ lie on a circle γ_1. The diagonals AC, CE, EB, BD, and DA are tangents to another circle γ_2 with the same centre as γ_1.

(a) Show that all angles of the pentagon $ABCDE$ have the same size and that all edges of the pentagon have the same length.

(b) What is the ratio of the radii of the circles γ_1 and γ_2? (The answer should be given in terms of integers, the four basic arithmetic operations and extraction of roots only.)

Problem 3

(a) Let x and y be two positive integers such that $\sqrt{x} + \sqrt{y}$ is an integer. Show that $\sqrt{x}$ and $\sqrt{y}$ are both integers.

(b) Find all positive integers x and y such that $\sqrt{x} + \sqrt{y} = \sqrt{2007}$.

Problem 4

Let a, b and c be integers such that $a + b + c = 0$.

(a) Show that $a^4 + b^4 + c^4$ is divisible by $a^2 + b^2 + c^2$.

(b) Show that $a^{100} + b^{100} + c^{100}$ is divisible by $a^2 + b^2 + c^2$.