English

The Niels Henrik Abels mathematics competition 2008–2009

Final round 12 March 2009

In the final round of the Abel contest there are 4 problems (8 subproblems) to be solved in 4 hours. You are required to show the reasoning behind your answers. Start a new sheet of paper for each problem.

The maximum score is 10 points for each problem. The total score is thus between 0 and 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

a. Show that there exist infinitely many integers that cannot be written as the difference between two perfect squares.

b. Show that the sum of three consecutive perfect cubes can always be written as the difference between two perfect squares.

(A perfect square is an integer raised to the second power. A perfect cube is an integer raised to the third power.)

Problem 2

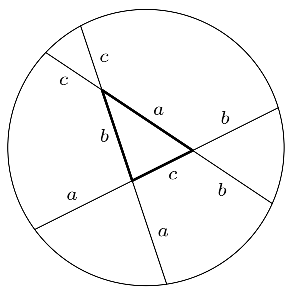
There are two letters in a language. Every word consists of seven letters, and two different words always have different letters on at least three places.

a. Show that such a language cannot have more than 16 words.

b. Can there be 16 words in the language?

Problem 3

a. In the triangle ABC the edge BC has length a, the edge AC length b, and the edge AB length c. Extend all the edges at both ends – by the length a from the vertex A, b from B, and c from C. Show that the six endpoints of the extended edges all lie on a common circle.



b. Show for any positive integer n that there exists a circle in the plane such that there are exactly n grid points within the circle. (A grid point is a point having integer coordinates.)

Problem 4

a. Show that $\left(\dfrac{2010}{2009}\right)^{2009} > 2$.

b. Let $x = 1 - 2^{-2009}$. Show that $x + x^2 + x^4 + x^8 + \cdots + x^{2^m} < 2010$ for all positive integers m.