English

The Niels Henrik Abels mathematics competition 2009–2010

Final round   11 March 2010

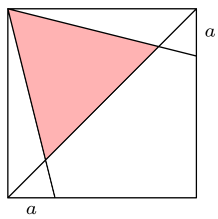
In the final round of the Abel contest there are four problems (eight subproblems) to be solved in four hours. You are required to show the reasoning behind your answers. Start a new sheet of paper for each problem.

The maximum score is 10 points for each problem. The total score is thus between 0 and 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

a. The point P lies on the edge AB of a quadrilateral $ABCD$. The angles BAD, ABC and CPD are right, and $AB = BC + AD$. Show that $BC = BP$ or $AD = BP$.



b. The edges of the square in the figure have length 1. Find the area of the marked region in terms of a, where $0 \le a \le 1$.

Problem 2

a. Show that $\dfrac{x^2}{1-x} + \dfrac{(1-x)^2}{x} \ge 1$ for all real numbers x, where $0 < x < 1$.

b. Show that $abc \le (ab+bc+ca)(a^2+b^2+c^2)^2$ for all positive real numbers a, b and c such that $a+b+c=1$.

Problem 3

a. There are 25 participants in a mathematics contest having four problems. Each problem is considered solved or not solved (that is, partial solutions are not possible). Show that either there are four contestants having solved the same problems (or not having solved any of them), or two contestants, one of which has solved exactly the problems that the other did not solve.

b. There are k sport clubs for the students of a secondary school. The school has 100 students, and for any selection of three of them, there exists a club having at least one of them, but not all, as a member. What is the least possible value of k?

Problem 4

a. Find all positive integers k and l such that $k^2 - l^2 = 1005$.

b. Let $n > 2$ be an integer. Show that it is possible to choose n points in the plane, not all of them lying on the same line, such that the distance between any pair of points is an integer (that is, $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is an integer for all pairs (x_1, y_1) and (x_2, y_2) of points).