English

Niels Henrik Abel Mathematics Competition 2010–2011

Final 10 March 2011

In the final round of the Abel contest there are four problems (eight subproblems) to be solved in four hours. You are required to justify your answers. Start a new sheet of paper for each problem.

The maximum score is 10 points for each problem. The maximum score is thus 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

Let n be the number that is produced by concatenating the numbers 1, 2, ..., 4022, that is, $n = 1234567891011\ldots40214022$.

a. Show that n is divisible by 3.

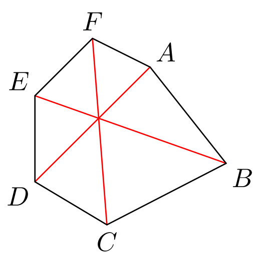
We consider the sum of the digits of a positive integer. For example, the sum of the digits of 2536 is 16, since $2 + 5 + 3 + 6 = 16$.

b. Let $a_1 = n^{2011}$, and let a_i be the sum of the digits of a_{i-1} for $i > 1$. Find a_4.

Problem 2

a. In the quadrilateral $ABCD$ the side AB has length 7, BC length 14, CD length 26, and DA length 23. Show that the diagonals are perpendicular. You may assume that the quadrilateral is convex (all internal angles are less than 180°).

b. The diagonals AD, BE, and CF of a convex hexagon $ABCDEF$ intersect in a common point. Show that $a(ABE)\,a(CDA)\,a(EFC) = a(BCE)\,a(DEA)\,a(FAC)$, where $a(KLM)$ is the area of the triangle KLM.



Problem 3

a. The positive numbers $a_1, a_2, \ldots$ satisfy $a_1 = 1$ and $(m+n)a_{m+n} \le a_m + a_n$ for all positive integers m and n. Show that $1/a_{200} > 4 \cdot 10^7$.

b. Find all functions f from the real numbers to the real numbers such that $f(xy) \le \frac{1}{2}\bigl(f(x) + f(y)\bigr)$ for all real numbers x and y.

Problem 4

a. In a town there are n avenues running from south to north. They are numbered 1 through n (from west to east). There are n streets running from west to east – they are also numbered 1 through n (from south to north). If you drive through the junction of the kth avenue and the lth street, you have to pay kl kroner. How much do you at least have to pay for driving from the junction of the 1st avenue and the 1st street to the junction of the nth avenue and the nth street? (You also pay for the starting and finishing junctions.)

b. In a group of 199 persons, each person is a friend of exactly 100 other persons in the group. All friendships are mutual, and we do not count a person as a friend of himself/herself. For which integers $k > 1$ is the existence of k persons, all being friends of each other, guaranteed?