English

The Niels Henrik Abel mathematics competition 2012–2013

Final 7 March 2013

In the final round of the Abel contest there are four problems (six subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

The maximum score is 10 points for each problem. The maximum score is thus 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

a. Find all real numbers a such that the inequality

$$3x^2 + y^2 \geq -ax(x+y)$$

holds for all real numbers x and y.

b. The sequence a_1, a_2, a_3, … is defined so that $a_1 = 1$ and

$$a_{n+1} = \frac{a_1 + a_2 + \cdots + a_n}{n} + 1 \qquad \text{for } n \geq 1.$$

Show that for every positive real number β we can find a k so that $a_k < \beta k$.

Problem 2

In a triangle T, all the angles are less than 90°, and the longest side has length s. Show that for every point p in T we can pick a corner h in T such that the distance from p to h is less than or equal to $s/\sqrt{3}$.

Problem 3

A prime number $p \geq 5$ is given. Write

$$\frac{1}{3} + \frac{2}{4} + \cdots + \frac{p-3}{p-1} = \frac{a}{b}$$

for natural numbers a and b. Show that p divides a.

Problem 4

a. An ordered quadruple (P_1, P_2, P_3, P_4) of corners in a regular 2013-gon is called *crossing* if the four corners are all different, and the line segment from P_1 to P_2 intersects the line segment from P_3 to P_4. How many crossing quadruples are there in the 2013-gon?

b. A total of $a \cdot b \cdot c$ cubical boxes are joined together in a $a \times b \times c$ rectangular stack, where $a, b, c \geq 2$. A bee is found inside one of the boxes. It can fly from one box to another through a hole in the wall, but not through edges or corners. Also, it cannot fly outside the stack. For which triples (a, b, c) is it possible for the bee to fly through all of the boxes exactly once, and end up in the same box where it started?