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English

The Niels Henrik Abel mathematics competition 2015–2016

Final   1 March 2016

In the final round of the Abel contest there are four problems (six subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

You can score up to 10 points for each problem. The maximum score is thus 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

A *walking sequence* is a sequence of integers with $a_{i+1} = a_i \pm 1$ for every i. Show that there exists a sequence $b_1, b_2, \ldots, b_{2016}$ such that for every walking sequence $a_1, a_2, \ldots, a_{2016}$ where $1 \le a_i \le 1010$, there is for some j for which $a_j = b_j$.

Problem 2

a. Find all positive integers a, b, c, d with $a \le b$ and $c \le d$ such that

$$a + b = cd,$$
$$c + d = ab.$$

b. Find all non-negative integers x, y and z such that

$$x^3 + 2y^3 + 4z^3 = 9!.$$

(Here $9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, as usual.)

Problem 3

a. Three circles S_A, S_B, and S_C in the plane with centers in A, B, and C, respectively, are mutually tangential on the outside. The point of tangency between S_A and S_B we call C', the one S_A between S_C we call B', and the one between S_B and S_C we call A'. The common tangent between S_A and S_C (passing through B') we call ℓ_B, and the common tangent between S_B and S_C (passing through A') we call ℓ_A. The point of intersection of ℓ_A and ℓ_B is called X. The point Y is located so that $\angle XBY$ and $\angle YAX$ are both right angles. Show that the points X, Y, and C' lie on a line if and only if $AC = BC$.

b. Let ABC be an acute triangle with $AB < AC$. The points A_1 and A_2 are located on the line BC so that AA_1 and AA_2 are the inner and outer angle bisectors at A for the triangle ABC. Let A_3 be the mirror image A_2 with respect to C, and let Q be a point on AA_1 such that $\angle A_1QA_3 = 90°$. Show that $QC \parallel AB$.

Problem 4

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the equation

$$f(x)f(y) = |x-y| \cdot f\left(\frac{xy+1}{x-y}\right)$$

holds for all choices of two different real numbers x and y.