ABEL KONKURRANSEN

English

The Niels Henrik Abel mathematics competition 2016–2017

Final 7 March 2017

In the final round of the Abel contest there are four problems (six subproblems) to be solved in four hours. You are required to justify your answers. **Start a new sheet of paper for each of the four problems.**

You can score up to 10 points for each problem. The maximum score is thus 40.

No aids other than writing paper, writing tools and bilingual dictionaries are permitted.

Problem 1

a. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x)f(y) = f(xy) + xy$$

for all $x, y \in \mathbb{R}$.

b. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x)f(y) = f(x + y) + xy$$

for all $x, y \in \mathbb{R}$.

Problem 2

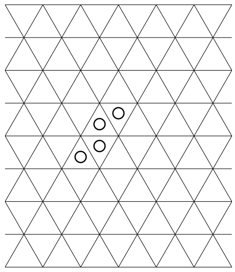
Let the sequence a_n be defined by $a_0 = 2$, $a_1 = 15$, and $a_{n+2} = 15a_{n+1} + 16a_n$ for $n \geq 0$. Show that there are infinitely many integers k such that $269 \mid a_k$.

Problem 3

a. Nils has a telephone number with eight different digits. He has made 28 cards with statements of the type “The digit a occurs earlier than the digit b in my telephone number” – one for each pair of digits appearing in his number.

How many cards can Nils show you without revealing his number?

b. In an infinite grid of regular triangles, Niels and Henrik are playing a game they made up. Every other time, Niels picks a triangle and writes $\times$ in it, and every other time, Henrik picks a triangle where he writes a $\circ$. If one of the players gets four in a row in some direction (see figure), he wins the game.



Determine whether one of the players can force a victory, or if both players can stop the other from winning.

Problem 4

Let $a > 0$ og $0 < \alpha < \pi$ be given. Let ABC be a triangle with $BC = a$ and $\angle BAC = \alpha$, and call the cicumcentre O, and the orthocentre H. The point P lies on the ray from A through O. Let S be the mirror image of P through AC, and T the mirror image of P through AB. Assume that $SATH$ is cyclic. Show that the length AP depends only on a and α.