The Niels Henrik Abel Contest 1998–99

FINAL

March 12th 1999

Problem 1

a) Determine a function $f(x)$ such that $f(t^2+t+1)=t$ for all real numbers $t \geq 0$.

b) Prove that for all real numbers a, b, c, d and e, the ineuqality

$$a^2+b^2+c^2+d^2+e^2 \geq a(b+c+d+e)$$

applies.

Problem 2

a) Find all integers m and n such that $2m^2+n^2=2mn+3n$.

b) Given positive integers a, b and c such that a^3 is divisible by b, b^3 is divisible by c, and c^3 is divisible by a, prove that $(a+b+c)^{13}$ is divisible by abc.

Problem 3

Let ΔABC be an isosceles triangle with $AB = AC$ and $\angle A = 30°$. The triangle is inscribed in a circle with centre O. The point D lies on the arch between A and C such that $\angle DOC = 30°$. Let G be the point on the arch between A and B such that $DG = AC$ and $AG < BG$. The line DG intersects AC and AB in E and F respectively.

a) Prove that ΔAFG is equilateral.

b) Find the ratio between the areas $\Delta AFE/\Delta ABC$.

Problem 4

Let S be the set $\{1, 2, 3, \ldots, 10\}$. For every non-empty subset R of S, we define the alternating sum $A(R)$ in the following way: If $\{r_1, r_2, \ldots, r_k\}$ are the elements of R ordered in increasing order, the alternating sum is $A(R) = r_k - r_{k-1} + r_{k-2} - \cdots - (-1)^k r_1$, where $+$ and $-$ comes alternatingly. Eg., the alternating sum of $\{1, 3, 4, 7\}$ is $7-4+3-1=5$.

a) Is it possible to write S as a union of two non-intersecting sets having the same alternating sum?

b) Let $\{R_1, R_2, \ldots, R_n\}$ be the set of all non-empty subsets of S. Determine the sum $A(R_1)+A(R_2)+\cdots+A(R_n)$.