

Eighth International Olympiad, 1966

1966/1.

In a mathematical contest, three problems, A, B, C were posed. Among the participants there were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A , the number who solved B was twice the number who solved C . The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A . How many students solved only problem B ?

1966/2.

Let a, b, c be the lengths of the sides of a triangle, and α, β, γ , respectively, the angles opposite these sides. Prove that if

$$a + b = \tan \frac{\gamma}{2}(a \tan \alpha + b \tan \beta),$$

the triangle is isosceles.

1966/3.

Prove: The sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

1966/4.

Prove that for every natural number n , and for every real number $x \neq k\pi/2^t$ ($t = 0, 1, \dots, n; k$ any integer)

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \dots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x.$$

1966/5.

Solve the system of equations

$$\begin{array}{rcccc} & & |a_1 - a_2| x_2 & + & |a_1 - a_3| x_3 & + & |a_1 - a_4| x_4 & = & 1 \\ |a_2 - a_1| x_1 & & & + & |a_2 - a_3| x_3 & + & |a_2 - a_4| x_4 & = & 1 \\ |a_3 - a_1| x_1 & + & |a_3 - a_2| x_2 & & & & & = & 1 \\ |a_4 - a_1| x_1 & + & |a_4 - a_2| x_2 & + & |a_4 - a_3| x_3 & & & = & 1 \end{array}$$

where a_1, a_2, a_3, a_4 are four different real numbers.

1966/6.

In the interior of sides BC, CA, AB of triangle ABC , any points K, L, M , respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of triangle ABC .