Eighteenth International Olympiad, 1976

1976/1.

In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.

1976/2.

Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \cdots$. Show that, for any positive integer n, the roots of the equation $P_n(x) = x$ are real and distinct.

1976/3.

A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, so that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of all such boxes.

1976/4.

Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

1976/5.

Consider the system of p equations in $q = 2p$ unknowns $x_1, x_2, \cdots, x_q$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1q}x_q &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2q}x_q &= 0 \\ &\cdots \\ a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pq}x_q &= 0 \end{aligned}$$

with every coefficient a_{ij} member of the set $\{-1, 0, 1\}$. Prove that the system has a solution $(x_1, x_2, \cdots, x_q)$ such that
(a) all x_j $(j = 1, 2, ..., q)$ are integers,
(b) there is at least one value of j for which $x_j \neq 0$,
(c) $|x_j| \leq q(j = 1, 2, ..., q)$.

1976/6.

A sequence $\{u_n\}$ is defined by

$$u_0 = 2, u_1 = 5/2, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1 \text{for } n = 1, 2, \cdots$$

Prove that for positive integers n,

$$[u_n] = 2^{[2^n-(-1)^n]/3}$$

where $[x]$ denotes the greatest integer $\leq x$.