34th IMO 1993

A1. Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the produce of two non-constant polynomials with integer coefficients.

A2. Let D be a point inside the acute-angled triangle ABC such that $\angle ADB = \angle ACB + 90^o$, and $AC \cdot BD = AD \cdot BC$.
(a) Calculate the ratio $AB \cdot CD/(AC \cdot BD)$.
(b) Prove that the tangents at C to the circumcircles of ACD and BCD are perpendicular.

A3. On an infinite chessboard a game is played as follows. At the start n^2 pieces are arranged in an $n \times n$ block of adjoining squares, one piece on each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of n for which the game can end with only one piece remaining on the board.

B1. For the points P, Q, R in the plane define $m(PQR)$ as the minimum length of the three altitudes of the triangle PQR (or zero if the points are collinear). Prove that for any points A, B, C, X:

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

B2. Does there exist a function f from the positive integers to the positive integers such that $f(1) = 2, f(f(n)) = f(n) + n$ for all n, and $f(n) < f(n+1)$ for all n?

B3. There are $n > 1$ lamps $L_0, L_1, \dots, L_{n-1}$ in a circle. We use L_{n+k} to mean L_k. A lamp is at all times either on or off. Initially they are all on. Perform steps $s_0, s_1, \dots$ as follows: at step s_i, if L_{i-1} is lit, then switch L_i from on to off or vice versa, otherwise do nothing. Show that:
(a) There is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are on again;
(b) If $n = 2^k$, then we can take $M(n) = n^2 - 1$;
(c) If $n = 2^k + 1$, then we can take $M(n) = n^2 - n + 1$.

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