35th IMO 1994

A1. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k. Prove that:

$$(a_1 + \ldots a_m) \geq \frac{(n+1)}{2}.$$

A2. ABC is an isosceles triangle with $AB = AC$, M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB. Q is an arbitrary point on BC different from B and C. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear Prove that OQ is perpendicular to EF iff $QE = QF$.

A3. For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ which have exactly three 1s when written in base 2. Prove that for each positive integer m, there is at least one k with $f(k) = m$, and determine all m for which there is exactly one k.

B1. Determine all ordered pairs (m, n) of positive integers for which $\frac{n^3+1}{mn-1}$ is an integer.

B2. Let S be the set of all real numbers greater than -1. Find all functions $f\colon S \to S$ such that $f\big(x + f(y) + xf(y)\big) = y + f(x) + yf(x)$ for all x, y, and $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

B3. Show that there exists a set A of positive integers with the following property: for any infinite set S of primes, there exist two positive integers $m \in A$ and $n \notin A$, each of which is a product of k distinct elements of S for some $k \geq 2$.

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25 August 2003