

First day

Mar del Plata, Argentina - July 24, 1997

Problem 1

In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard).

For any pair of positive integers m and n , consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n , lie along edges of the squares.

Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let

$$f(m, n) = |S_1 - S_2|.$$

a) Calculate $f(m, n)$ for all positive integers m and n which are either both even or both odd.

b) Prove that $f(m, n) \leq \frac{1}{2} \max\{m, n\}$ for all m and n .

c) Show that there is no constant C such that $f(m, n) < C$ for all m and n .

Problem 2

Angle A is the smallest in the triangle ABC .

The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A .

The perpendicular bisectors of AB and AC meet the line AU at V and W , respectively. The lines BV and CW meet at T .

Show that

$$AU = TB + TC.$$

Problem 3

Let x_1, x_2, \dots, x_n be real numbers satisfying the conditions:

$$|x_1 + x_2 + \dots + x_n| = 1$$

and

$$|x_i| \leq \frac{n+1}{2} \quad \text{for } i = 1, 2, \dots, n.$$

Show that there exists a permutation y_1, y_2, \dots, y_n of x_1, x_2, \dots, x_n such that

$$|y_1 + 2y_2 + \dots + ny_n| \leq \frac{n+1}{2}.$$

Second day

Mar del Plata, Argentina - July 25, 1997

Problem 4

An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \dots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, \dots, n$, the i th row and the i th column together contain all elements of S . Show that

- a) there is no silver matrix for $n = 1997$;
- b) silver matrices exist for infinitely many values of n .

Problem 5

Find all pairs (a, b) of integers $a \geq 1, b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.$$

Problem 6

For each positive integer n , let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents.

Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$ because the number 4 can be represented in the following four ways: 4; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1.

Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$