14th Nordic Mathematical Contest
Solutions

Problem 1. Set x = the number of sums with 3 different integers, y = the numbers of sums with 2 different integers. Consider a sequence of 3999 numbered boxes where every odd-numbered box contains a red ball. Every placement of blue balls in any two of the even-numbered boxes produces a division of 2000 in three parts. There are $\binom{1999}{2} = 999 \cdot 1999$ ways of placing the two balls. Now every division of 2000 in three parts of different size is produced by $3! = 6$ different placements, and every division having two equal parts is produced by 3 different placements. So $6x + 3y = 1999 \cdot 999$. But $y = 999$, since the two equal parts can assume any size from 1 to 999. Solving, we get $x = 998 \cdot 333$, $x + y = 1001 \cdot 333 = 333333$.

Problem 2. Assume P_n originally has m coins, P_{n-1} $m+1$ coins, $\ldots$, P_1 $m+n-1$ coins. In every move, a person receives k coins and gives $k+1$ coins, so in total his fortune diminishes by one coin. After the first round, when P_n has left n coins to P_1, P_n has $m-1$ coins, P_{n-1} has m coins, etc., after two rounds P_n has $m-2$ coins, P_{n-1} has $m-1$ coins, etc. We can continue like this for m rounds, and after that P_n has no money, P_{n-1} has one coin etc. Now in round $m+1$ every person who has money can receive money and give away money as before, except P_n who was bankrupt. He receives $n(m+1)-1$ coins from P_{n-1}, but cannot give $n(m+1)$ coins away. In this situation P_{n-1} has one coin, and P_1 has $n-2$ coins. The only pair of neighbors where one player can have 5 times as many coins as the other is (P_1, P_n). Because $n - 2 < n(m+1) - 1$, we must have $5(n-2) = n(m+1) - 1$ or $n(4-m) = 9$. Since $n > 1$, either $n = 3$, $m = 1$ or $n = 9$, $m = 3$. Both alternatives work: in the first case, the number of coins is $3 + 2 + 1 = 6$, in the second, $11 + 10 + \cdots + 3 = 63$.

Problem 3, solution 1. Consider triangles AOE and AOD. They have two equal sides, and equal angles opposite to one pair of equal sides. Then either AOE and AOD are congruent or $\angle AEO = 180° - \angle ADO$. In the first case, $\angle BEO = \angle CDO$, and the triangles EBO and DCO are congruent. Altogether, then, $AB = AC$. In the second case, denote the angles at A, B, and C by 2α, 2β, and 2γ, respectively, and $\angle AEO$ by δ. Using the theorem of the adjacent angle in a triangle,we get $\angle BOE = \angle DOC = \beta + \gamma$, $\delta = 2\beta + \gamma$, $180° - \delta = \beta + 2\gamma$. Adding these,we have $3(\beta + \gamma) = 180°$, $\beta + \gamma = 60°$. Combining this with $2(\alpha + \beta + \gamma) = 180°$, we get $2\alpha = 60°$.

Problem 3, solution 2. Let β, γ be as above. Using the sine theorem in $\triangle BEO$ and $\triangle CDO$, we obtain

$$\frac{OE}{\sin\beta} = \frac{OB}{\sin(180° - 2\beta - \gamma)}, \quad \frac{OD}{\sin\gamma} = \frac{OC}{\sin(180° - \beta - 2\gamma)}.$$

These combine to

$$\frac{OB}{OC} = \frac{\sin(2\beta + \gamma)\sin\gamma}{\sin(\beta + 2\gamma)\sin\beta}.$$

Using the theorem of sines to $\triangle BOC$, we obtain

$$\frac{OB}{OC} = \frac{\sin\gamma}{\sin\beta}.$$

So we must have $\sin(\beta + 2\gamma) = \sin(2\beta + \gamma)$. So either $\beta + 2\gamma = 2\beta + \gamma$ or $\beta + 2\gamma = 180^\circ - 2\beta - \gamma$. The first equation implies $\beta = \gamma$, or the isosceles case, while the second one gives $\beta + \gamma = 60^\circ$, which easily leads to $\angle BAC = 60^\circ$.

Problem 4. Assuming $0 \le x < y < z \le 1$ and $y - x = z - y$, we have

$$\begin{aligned} f(z) - f(y) &\le 2f(y) - 2f(x) \\ f(y) - f(x) &\le 2f(z) - 2f(z) - f(y), \end{aligned}$$

or

$$\frac{2}{3}f(x) + \frac{1}{3}f(z) \le f(y) \le \frac{1}{3}f(x) + \frac{2}{3}f(z). \tag{1}$$

Denote $f\left(\frac{1}{3}\right)$ by a and $f\left(\frac{2}{3}\right)$ by b. Apply (1) with $x = 0$, $y = \dfrac{1}{3}$, $z = \dfrac{2}{3}$, and $x = \dfrac{1}{3}$, $y = \dfrac{2}{3}$, and $z = 1$, to obtain

$$\begin{aligned} \frac{1}{3}b \le a \le \frac{2}{3}b \\ \frac{2}{3}a + \frac{1}{3} \le b \le \frac{1}{3}a + \frac{2}{3}. \end{aligned}$$

Eliminating b, we have

$$\frac{1}{3}\left(\frac{2}{3}a + \frac{1}{3}\right) \le a \le \frac{2}{3}\left(\frac{1}{3}a + \frac{2}{3}\right),$$

fron which one solves for a to obtain $\dfrac{1}{7} \le a \le \dfrac{4}{7}$. – In fact, the bounds cannot be reached; one can show that the sharp bounds for $f(\frac{1}{3})$ are $\dfrac{4}{27}$ and $\dfrac{76}{135}$.