15th Nordic Mathematical Contest

Thursday March 29th, 2001

English version

Time allowed: 4 hours. Each problem is worth 5 points.

Problem 1. Let A be a finite collection of squares in a coordinate plane such that each square in A has for its corners points of the form (m, n), $(m+1, n)$, $(m, n+1)$ and $(m+1, n+1)$ for some integers m and n.

Show that there exists a subcollection B of A consisting of at least 25% of all the squares in A such that no two distinct squares in B have a common corner point.

Problem 2. Let f be a bounded real-valued function defined for all real values such that the following condition is satisfied for every real number x:

$$f(x + \tfrac{1}{3}) + f(x + \tfrac{1}{2}) = f(x) + f(x + \tfrac{5}{6})$$

Show that f is periodic. (A function f is called periodic, if there exists a positive number k, such that $f(x+k) = f(x)$ for every real number x).

Problem 3. Determine the number of real roots in the equation

$$x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \tfrac{5}{2} = 0$$

Problem 4. Let $ABCDEF$ be a convex hexagon in which each of the diagonals AD, BE and CF divides the hexagon in two quadrilaterals with equal areas.

Show that AD, BE and CF pass through the same point.

Solution 1. Let G be the set of all the squares with corner points of the form (m,n), $(m+1,n)$, $(m,n+1)$ and $(m+1,n+1)$ for some integers m and n. A is a subset of G. Let's assign to each of the squares in G one af the numbers 1, 2, 3 and 4 in the following way: In every second row we assign the squares alternately the integers 1 and 2. The squares just below the squares with a 1-integer we assign the integer 3. The rest of the squares we assign the integer 4.

Let A_i be the subset of A containing the squares numbered i $(i=1,2,3,4)$. No two distinct squares of the set A_i have a common corner. Hence the biggest of the sets A_1, A_2, A_3 and A_4 can be choosen as the set B.

Solution 2. We use the condition $f(x)=f(x+\frac{1}{3})+f(x+\frac{1}{2})-f(x+\frac{5}{6})$ several times for different x

$$\begin{aligned}
&f(x)=f(x+\tfrac{1}{3})+f(x+\tfrac{1}{2})-f(x+\tfrac{5}{6})\\
&=(f(x+\tfrac{2}{3})+f(x+\tfrac{5}{6})-f(x+\tfrac{7}{6}))+(f(x+\tfrac{5}{6})+f(x+1)-f(x+\tfrac{4}{3}))-f(x+\tfrac{5}{6})\\
&=(f(x+1)+f(x+\tfrac{7}{6})-f(x+\tfrac{3}{2}))+f(x+\tfrac{5}{6})-f(x+\tfrac{7}{6})+f(x+1)-f(x+\tfrac{4}{3})\\
&=2f(x+1)-f(x+\tfrac{3}{2})+(f(x+\tfrac{7}{6})+f(x+\tfrac{4}{3})-f(x+\tfrac{5}{3}))-f(x+\tfrac{4}{3})\\
&=2f(x+1)-f(x+\tfrac{3}{2})+(f(x+\tfrac{3}{2})+f(x+\tfrac{5}{3})-f(x+2))-f(x+\tfrac{5}{3})\\
&=2f(x+1)-f(x+2)
\end{aligned}$$

Hence

$$f(x+2)-f(x+1)=f(x+1)-f(x)$$

Telescoping now gives

$$f(x+n)-f(x)=\sum_{i=1}^{n}((f(x+i)-f(x+i-1))=n(f(x+1)-f(x))$$

From this we see that f can be bounded only if $f(x+1)-f(x)=0$ i.e. f is periodic with a period of 1.

Solution 3. $x^8-x^7+2x^6-2x^5+3x^4-3x^3+4x^2-x+\frac{5}{2}=x(x-1)(x^6+2x^4+3x^2+4)+\frac{5}{2}$. Only if $0<x<1$ is $x(x-1)$ negative and that's neccessary, if x is a root in the equation. $x(x-1)\geq -\frac{1}{4}$ and for $0<x<1$ we have $x(x-1)(x^6+2x^4+3x^2+4)+\frac{5}{2}>-\frac{1}{4}(1+2+3+4)+\frac{5}{2}=0$. Thus the equation has no real roots.

Solution 4. Assume that the three diagonals AD, BE and DF do not have any point in common. Let AD and BE intersect in R, BE and CF intersect in P, CF and AD intersect in Q. The points R and Q divide the diagonal AD in three pieces a, d and g. g is the linesegment RQ and a is the linesegment with endpoint R. The points P and R divide the diagonal BE in three pieces b, e and h. h is the linesegment PR and b is the linesegment with endpoint R. Similarly the points P and Q divide the diagonal CF in three pieces c, f and j. j is the linesegment PQ and c is the linesegment with endpoint Q.
We now look more closely on the areas

$$a(ARB) = \frac{1}{2}a(ABCDEF) - a(BCDR) = a(DRE)$$

Hence

$$ab = (d+g)(e+h)$$

Similarly we get

$$cd = (f+j)(a+g)$$

$$ef = (b+h)(c+j)$$

Multiplying these equalities we get

$$abcdef = (a+g)(b+h)(c+j)(d+g)(e+h)(f+j)$$

Since a, b, c, d, e and f are positive and g, h and j are non-negative numbers, every factor on the left side is less or equal than the corresponding factor on the right side. Hence $g = h = j = 0$. Thus the three diagonals pass through the same point.