The 17$^{\text{th}}$ Nordic Mathematical Contest

Thursday April 3rd 2003

Problem 1

We place some stones on a rectangular chessboard with 10 rows and 14 columns. Afterwards we realize that there are an odd number of stones in each row and each column. If the squares of the chessboard are coloured alternatingly black and white (as is usual), show that the total number of stones on black squares is even. Note that a square may contain more than one stone.

Problem 2

Find all triplets (x, y, z) of integers such that

$$x^3 + y^3 + z^3 - 3xyz = 2003.$$

Problem 3

The equilateral triangle $\triangle ABC$ contains a point D such that $\angle ADC = 150°$. Prove that a triangle whose sides are $|AD|$, $|BD|$ and $|CD|$ is necessarily a right triangle.

Problem 4

Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ be the set of all real numbers except zero. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ satisfying

$$f(x) + f(y) = f(xyf(x + y))$$

for all $x, y \in \mathbb{R}^*$ such that $x + y \neq 0$.

Time permitted: 4 hours.
Each problem is worth 5 points.
Only writing and drawing utensils are allowed.

Solution 1

We number the rows from 1 to 10 and the columns from 1 to 14. We may assume that the square $(1,1)$ is white, so that a square (i,j) is black if and only if $i+j$ is odd. Now, the number of stones in the odd rows is a sum of 5 odd numbers, hence an odd number. Similarly, the number of stones in the odd columns is also an odd number. The sum of these two numbers is an even number, and it is also equal to the number of stones on the black squares plus two times the number of stones on the white squares (i,j) where both i and j are odd numbers. Hence the total number of stones on black squares is an even number.

Solution 2

By factorizing the left-hand side, we get $(x+y+z)(x^2+y^2+z^2-xy-xz-yz)=2003$. Since $x^2+y^2+z^2-xy-xz-yz=\frac{1}{2}((x-y)^2+(y-z)^2+(z-x)^2)$, both factors must be positive integers. Note that 3 cannot divide $x+y+z$, which implies that $x^2+y^2+z^2-xy-xz-yz=(x+y+z)^2-3(xy+xz+yz)\equiv 1 \mod 3$. Since 2003 is a prime number, and $2003 \equiv 2 \mod 3$, we must have $x+y+z=2003$. Hence $(x-y)^2+(y-z)^2+(z-x)^2=2$, and it follows that one of the terms must be zero. By symmetry we may assume that $x=y=z\pm 1$. Since $2003\equiv 2 \mod 3$, only $x=y=z+1$ works. This gives $(x,y,z)=(668,668,667)$, and the only solutions are the three permutations of this one.

Solution 3

Rotate $\triangle BCD$ leaving C fixed and moving B to A. The point D becomes a new point E. Since $|CE|=|CD|$ and $\angle DCE=60°$, triangle $\triangle CDE$ is equilateral. This implies that $\angle ADE=90°$, hence $\triangle ADE$ is a right triangle with side lengths $|AD|$, $|AE|=|BD|$ and $|DE|=|CD|$.

Solution 4

Take any $x\in\mathbb{R}^*$. For all $y\in\mathbb{R}^*$ with $y\neq x$, we have $f(y)+f(x-y)=f(y(x-y)f(x))$. The second degree equation $x-y=y(x-y)f(x)$ has the two solutions $y=x$ and $y=\frac{1}{f(x)}$. (Remember that $f(x)\neq 0$.) Now, if $f(x)\neq\frac{1}{x}$, we may let $y=\frac{1}{f(x)}\neq x$, which gives $f(y)=0$, a contradiction. Therefore, $f(x)=\frac{1}{x}$ for all $x\in\mathbb{R}^*$. It is easily verified that this actually is a solution, hence the only one.