The 31st Nordic Mathematical Contest

Monday, 3 April 2017

English version

Time allowed: 4 hours. Each problem is worth 7 points.
Only writing and drawing tools are allowed.

Problem 1 Let n be a positive integer. Show that there exist positive integers a and b such that:

$$\frac{a^2+a+1}{b^2+b+1} = n^2+n+1.$$

Problem 2 Let a, b, α, β be real numbers such that $0 \leq a, b \leq 1$, and $0 \leq \alpha, \beta \leq \frac{\pi}{2}$. Show that if

$$ab\cos(\alpha-\beta) \leq \sqrt{(1-a^2)(1-b^2)},$$

then

$$a\cos\alpha + b\sin\beta \leq 1 + ab\sin(\beta-\alpha).$$

Problem 3 Let M and N be the midpoints of the sides AC and AB, respectively, of an acute triangle ABC, $AB \neq AC$. Let ω_B be the circle centered at M passing through B, and let ω_C be the circle centered at N passing through C. Let the point D be such that $ABCD$ is an isosceles trapezoid with AD parallel to BC. Assume that ω_B and ω_C intersect in two distinct points P and Q. Show that D lies on the line PQ.

Problem 4 Find all integers n and m, $n > m > 2$, and such that a regular n-sided polygon can be inscribed in a regular m-sided polygon so that all the vertices of the n-gon lie on the sides of the m-gon.